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sebagai NARASUMBER dalam kegiatan **"Workshop on Graph Labelings"**

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# Workshop of Graph Labelings

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Universitas Merdeka Malang

Singaraja, August 28, 2019

# Graph Labeling

- $G$ is a finite and simple graph.

$$V(G) = \text{vertex set}; \ |V(G)| = p,$$
$$E(G) = \text{edge set}; \ |E(G)| = q.$$

- A *labeling* of a graph $G$ is a one to one mapping from some set of graph elements to a set of positive integers.
  - A *vertex labeling* $f : V(G) \to \{1, 2, 3, \ldots, p\}$.
  - An *edge labeling* $f : E(G) \to \{1, 2, 3, \ldots, q\}$.
  - A *total labeling* $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, p + q\}$.

# Weight

Let $f$ be a total labeling of $G$.

- *Vertex-weight* $w(v)$, $v \in V(G)$:
  Sum of label of $v$ and labels of its incident edges;

$$w(v) = f(v) + \sum_{u \in N(v)} f(uv).$$

- *Edge-weight* $w(e)$, $e = uv \in E(G)$:
  Sum of label of $e$ and of labels of its endpoints;

$$w(uv) = f(u) + f(uv) + f(v).$$

# Magic (Antimagic) Labeling

- A *vertex-magic* (*vertex-antimagic*) *total labeling*.
- An *edge-magic* (*edge-antimagic*) *total labeling*.

# Vertex-Magic Total Labeling

- A *vertex-magic total labeling* of a graph $G$ with $p$ vertices and $q$ is a bijective function

$$f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p + q\}$$

  such that $w(x) = f(x) + \sum_{y \in N(x)} f(xy) = k_f$ is a constant for any vertex $x \in V(G)$.
- $G$ is called a *vertex-magic total graph.*
- $k_f$ is called *magic constant* of $f$.

J.A. MacDougall, M. Miller, Slamin and W.D. Wallis, Vertex-magic total labelings of graphs , *Util. Math.*, **61** , (2002), 3 - 21.

# Vertex-Antimagic Total Labeling

- An $(a, d)$-*vertex-antimagic total labeling* of a graph $G$ with $p$ vertices and $q$ is a bijective function

$$f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p + q\}$$

  such that $\{f(x) + \sum_{y \in N(x)} f(xy) : x \in V(G)\} = \{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$ for two integers $a > 0$ and $d \geq 0$.
- $G$ is called an $(a, d)$-*vertex-antimagic total graph*.

Martin Baa, James A. MacDougall, Franois Bertault, Mirka Miller, Rinovia Simanjuntak, and Slamin, Vertex-antimagic total labelings of graphs , *Discus. Math. Graph Theory*, **23** (1), (2003), 67 - 83.

# Edge-Antimagic Total Labeling

- An $(a, d)$-*edge-antimagic total labeling* of a graph $G$ with $p$ vertices and $q$ is a bijective function

$$f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p + q\}$$

  such that $\{f(x) + f(xy) + f(y) : xy \in E(G)\} = \{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$ for two integers $a > 0$ and $d \geq 0$.
- $G$ is called an $(a, d)$-*edge-antimagic total graph.*

R. Simanjuntak, F. Bertault and M. Miller, Two new $(a, d)$-antimagic graph labelings , *Proc. of Eleventh Australian Workshop on Combinatorial Algorithm*, (2000), 179 - 184.

# Example

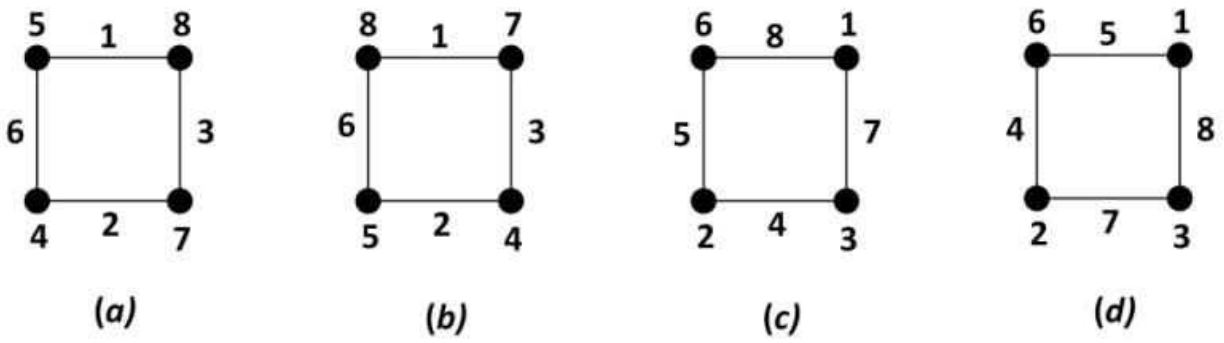


Figure: (a). A VMT graph with $k = 12$. (b). A (9, 2)-VAT graph. (c) A (9, 2)-EAT graph. (d) An EMT graph $k = 12$ .

# Edge-Magic Total Labeling

- An *edge-magic total labeling* (EMTL) of a graph $G$ with $p$ vertices and $q$ edges is a one to one mapping

$$f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p+q\}$$

  such that $f(x) + f(xy) + f(y) = k_f$ is a constant for any edge $xy$ of $G$.
- $G$ is called an *edge-magic total graph.*
- $k_f$ is called *magic constant* of $f$.

- **Conjecture**: Every tree is edge-magic total.

A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (4), (1970), 451–461.

## Super Edge-Magic Total Labeling

- An edge-magic total labeling $f$ of $G$ is called a *super edge-magic total labeling* (SEMTL) if $f(V(G)) = \{1, 2, 3, \cdots, p\}$.
- $G$ is called a *super edge-magic total graph*.

- **Conjecture**: Every tree is super edge-magic total.

H. Enomoto, A. Llado, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998), 105–109.

# Example

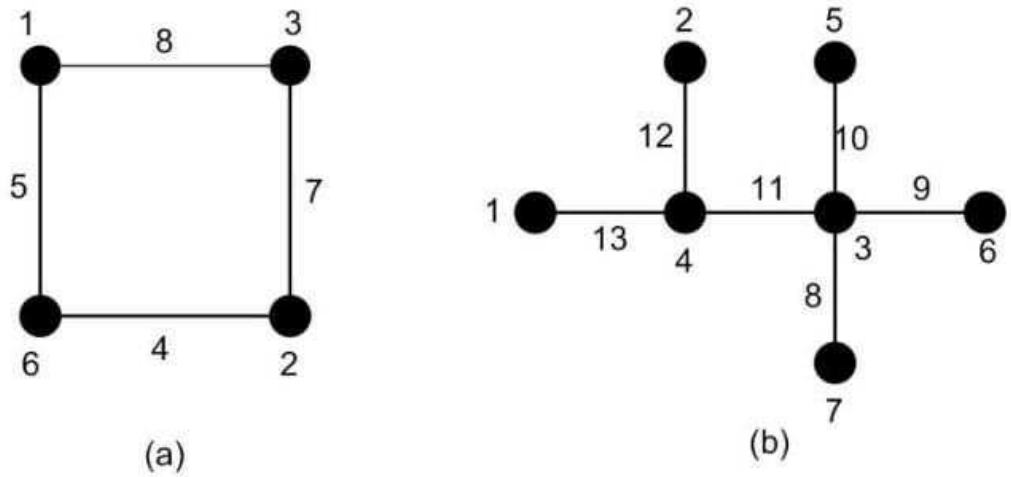


Figure: An edge-magic total graph with $k = 12$ and a super edge-magic total graph $k = 18$ .

# Elementary Counting

Let $G$ be a graph with $V(G) = \{x_1, x_2, \ldots, x_p\}$ and $f$ be an EMTL of $G$.

$$\sum_{xy \in E(G)} [f(x) + f(xy) + f(y)] = kq$$

This sum contains each edge label once, and each vertex label $d_i$ times, where $d_i$ is degree of the vertex $x_i$. Thus,

$$kq = \frac{1}{2}(p+q)(p+q+1) + \sum (d_i - 1) f(x_i) \ldots (*)$$

If $q$ is even, $p + q \equiv 2 \pmod 4$, and every $d_i$ is odd, then (*) is impossible.

## Elementary Counting

- If $G$ has $q$ even, $p + q \equiv 2 \pmod 4$, and every vertex has odd degree, then $G$ has no EMTL. [1]
  - The complete graph $K_n$ is not EMT when $n \equiv 4 \pmod 8$.
  - The Wheel $W_n$ is not EMT when $n \equiv 3 \pmod 4$.
  - The graph $tK_n$, consisting $t$ disjoin copies of $K_n$, is not EMT when $n \equiv 4 \pmod 8$ and $t$ is odd.
  - The Wheel $tW_n$ is not EMT when $n \equiv 3 \pmod 4$ and $t$ is odd.

**Reseach Problem.** [2]. Investigate graphs $G$ for which equation (*) implies the non-existence of an EMTL of $2G$.


[1]. G. Ringel and A. S. Llado, Another tree conjecture, *Bull. Inst. Combin. Appl.*, **18** , (1996), 83 – 85.

[2]. Alison M. Marr and W. D. Wallis, Magic Graphs, 2nd, Birkhäuser, Boston, 2013.

# Elementary Counting

Equation (*) may be used to provide bounds of magic constant $k$.

Let $d_1 \leq d_2 \leq \ldots \leq d_p$, then

$$kq \leq \lfloor \frac{1}{2}(p+q)(p+q+1) + d_1(q+1) + d_2(q+2) + \ldots + d_p(p+q) \rfloor$$

and

$$kq \geq \lceil \frac{1}{2}(p+q)(p+q+1) + d_1(p) + d_2(p+1) + \ldots + d_p(1) \rceil.$$

# Elementary Counting

**Example:** If $G = K_3$, then $k = 7 + \frac{1}{3}\sum_{i=1}^{3} f(x_i)$. So, $9 \leq k \leq 12$.

- $k = 9$, $\{f(x_1), f(x_2), f(x_3)\} = \{1, 2, 3\}$.
- $k = 10$, $\{f(x_1), f(x_2), f(x_3)\} = \{1, 3, 5\}$.
- $k = 11$, $\{f(x_1), f(x_2), f(x_3)\} = \{2, 4, 6\}$.
- $k = 12$, $\{f(x_1), f(x_2), f(x_3)\} = \{4, 5, 6\}$.

# Elementary Counting

**Example:** If $G = K_5$, then $k = 12 + \frac{3}{10}\sum_{i=1}^{5} f(x_i)$. So, $18 \leq k \leq 30$.

- $k = 18$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{1, 2, 3, 5, 9\}$.
- $k = 24$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{1, 8, 9, 10, 12\}$.
- $k = 24$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{4, 6, 7, 8, 15\}$.
- $k = 30$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{7, 11, 13, 14, 15\}$.
- $k = 21, 27$ no solutions.

## Dual Labeling

- If $f$ is an EMTL of a graph $G$ with magic constant $k$, then $f'(u) = (p+q+1) - f(u)$, $u \in V(G) \cup E(G)$, is an EMTL of $G$ with magic constant $k' = 3(p+q+1) - k$. [1]

- If $f$ is a SEMTL of a graph $G$ with magic constant $k$, then

$$f'(u) = \begin{cases} p+1-f(u), & \text{if } u \in v(g), \\ 2p+q+1-f(u), & \text{if } u \in E(G). \end{cases}$$

is a SEMTL of $G$ with magic constant $k' = 4p+q+3-k$. [2]


[1]. A. Kotzig and A. Rosa, Magic valuation of finite graphs, *Canad. Math. Bull.*, **13** (4), (1970), 451– 461.

[2]. R. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001), 153 – 168.

## Some Known EMT Graphs

- Cycle $C_n$ for any $n \geq 3$.
- Wheel $W_n$ for any $n \equiv 0, 1, 2 \pmod 4$.
- Fan $F_n$ for any $n \geq 2$.
- Sun $C_n \odot K_1$ for any $n \geq 3$.
- Complete bipartite graph $K_{n,m}$ for an $n \geq 1$ and $m \geq 1$.
- Book $B_{3,n}$, a graph consists of $n$ triangles with a common edge, for any $n \geq 1$.
- Generalize Petersen Graph $P(n, m)$ for odd $n \geq 3$.

**Research Problems:**

- The book $B_{m,n}$ consists of $n$ copies of $C_m$ with a common edge. Are all book $B_{m,n}$ EMT?
- An $(n, t)$-kite consists of a cycle $C_n$ with $t$-edge path attached to one vertex. Investigate the EMT properties of $(n, t)$-kites for any $t$.

## Some Known EMT Graphs

- If $G$ is a 3-colorable EMT graph, then $mG$ is EMT graph for every odd $m \geq 3$.

$3 \times m$ **array Kotzig is**

$$A = \begin{bmatrix} 0 & 1 & \dots & s-1 & s & s+1 & \dots & 2s-1 & 2s \\ s+1 & s+2 & \dots & 2s & 0 & 1 & \dots & s-1 & s \\ 2s-1 & 2s-3 & \dots & 1 & 2s & 2s-2 & \dots & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & \dots & s & s+1 & s+2 & \dots & 2s & 2s+1 \\ s+2 & s+3 & \dots & 2s+1 & 1 & 2 & \dots & s & s+1 \\ 2s & 2s-2 & \dots & 2 & 2s+1 & 2s-1 & \dots & 3 & 1 \end{bmatrix}$$

**Research Problems:**

- Is $mW_n$ an EMT graph when $n \equiv 1 \pmod 4$ and $m$ is odd?

A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (4), (1970), 451–461.

# (Super) Edge-Magic Total Strength

- The *(super) edge-magic total strength* of a graph $G$, $(s)emt(G)$, is the minimum of all magic constant $k$ where the minimum is taken over all (S)EMTL of $G$.That is,

$$(s)emt(G) = min\{k : f \text{ is an (S)EMTL of } G\}.$$

S. Avadayappan, P. Jeyanthi, and R. Vasuki, Magic strength of a graph , *Indian J. Pure Appl. Math.*, **31** (7), (2000), 873 – 883.

# Perfect (super) Edge-Magic Total Graph

- Let $T_G = \{\frac{\sum_{u\in V(G)} d(u)f(u)+\sum_{e\in E(G)} f(e)}{q} \mid f : V(G)\cup E(G) \longrightarrow \{1,2,\ldots,p+q\}$ is a bijection$\}$.
- (Super) magic interval of $G$, $J_G = \{\lceil \min T_G\rceil, \lceil \min T_G\rceil + 1, \ldots, \lfloor \max T_G\rfloor\}$.
- (Super) magic set of $G$, $\sigma_G = \{k \in J_G \mid k$ is magic constant of some (S)EMTL of $G\}$.
- $G$ is *perfect (S)EMT* if $J_G = \sigma_G$.


S. C. Lopez, F. A. Muntaner-Batle, and M. Rius-Font, Perfect super edge-magic graphs , *Bull. Math. Soc. Sci. Math. Roumanie Tome* , **55** (103), (2012), No. 2, 199 – 208.

S. C. Lopez, F. A. Muntaner-Batle, and M. Rius-Font, Perfect edge-magic graphs , *Bull. Math. Soc. Sci. Math. Roumanie Tome* , **57** (105), (2014), No. 1, 81 – 91.

# Super Edge-Magic Total Tree

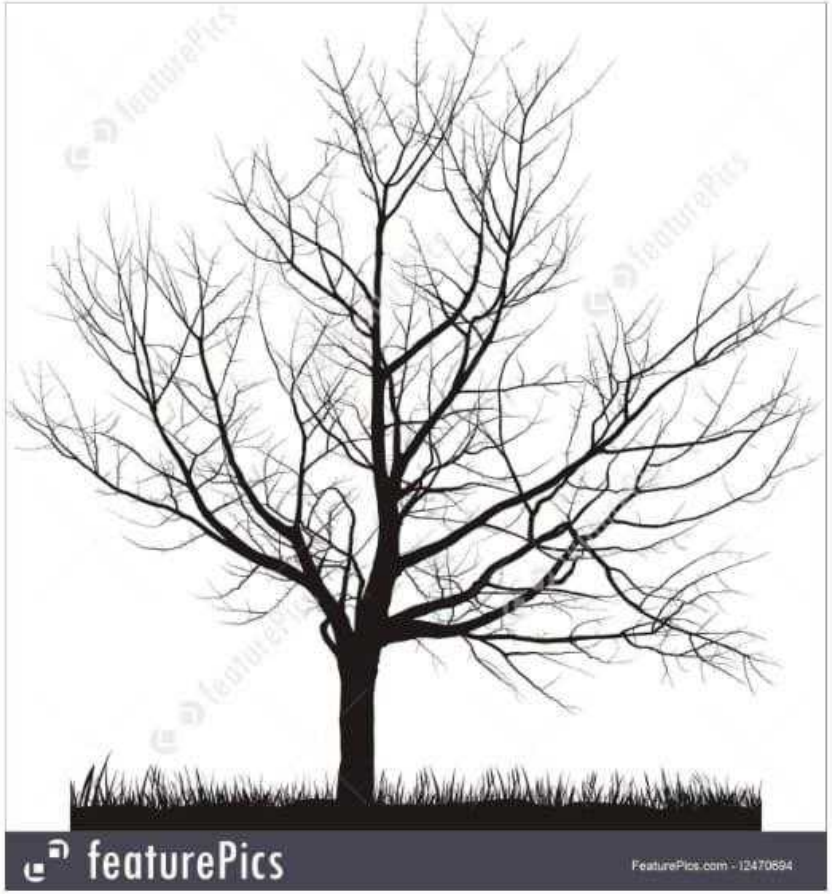


Figure: Is this a tree?

# Super Edge-Magic Total Tree

- Path
- Star
- Caterpillar
- Tree with at most 17 vertices
- Path-like-tree
- Symmetric binary tree
- Firecracker
- Banana tree

J. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **DS6** (2018) http://www.combinatorics.org.

# Super Edge-Magic Total Tree

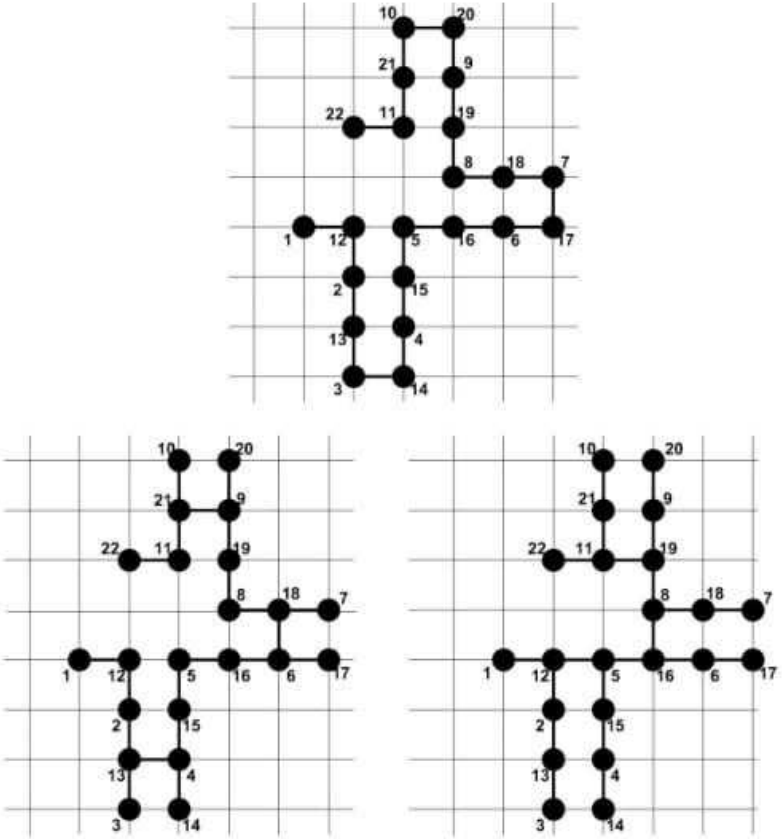


Figure: A path and path-like-trees?

# More Results on Super Edge-Magic Total Labeling

- If $G$ is SEMT, then $q \leq 2p - 3$. [1].
  - $W_n$ is not SEMT for any $n$.
  - $K_n$ is not SEMT for any $n \geq 3$.
  - No regular graph of degree greater than 3 can be SEMT.
  - Every SEMT graph contains at least two vertices of degree less than 4.

- A graph $G$ is SEMT if and only if there is a bijective function

$$f : V(G) \longrightarrow \{1, 2, 3, \ldots, p\}$$

such that the set $S = \{f(x) + f(y) | xy \in E(G)\}$ is a set of $q$ consecutive integers. [2].
  - If degree of every vertex of $G$ is even and $q \equiv 2 \pmod 4$, then is not a SEMT graph.

[1]. H. Enomoto, A. Llado, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998), 105–109.

[2]. R. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001), 153 – 168.

# More Results on Super Edge-Magic Total Labeling

- If a graph $G$ that is a tree or where $q \geq p$ is SEMT, then $G$ is sequential, harmonious, cordial. [1].
- Suppose that $G$ is a SEMT bipartite graph with partite sets $V_1$ and $V_2$ and let $f$ be a SEMTL of $G$ such that $f(V_1) = \{1, 2, 3, \ldots, |V_1|\}$, then $G$ has an $\alpha$-valuation. [1].


[1]. R. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001), 153 – 168.

[2]. J. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, **DS6** (2018) http://www.combinatorics.org.

# DISCUSSION

1. A sun $C_n \odot K_1$ is a graph constructed from a cycle $C_n$ by attaching one pendant to each vertex of the cycle $C_n$. Find the (super) edge-magic total strength of $C_3 \odot K_1$.
2. Prove that the sun $C_3 \odot K_1$ is a perfect (super) edge-magic total graph.
3. Find all possible values of magic constant for an edge-magic total labeling of $K_{1,m}$.
4. The graph $W_n - \{e\}$ is constructed from a wheel $W_n$ by deleting an edge. Which $W_n - \{e\}$ are super edge-magic total?

## Results on 2-regular Graphs

Holden et al. proved that,

- $C_5 \cup (2t)C_3$ is SEMT for each integer $t \geq 3$.
- $C_4 \cup (2t-1)C_3$ is SEMT for each integer $t \geq 3$.
- $C_7 \cup (2t)C_3$ is SEMT for each integer $t \geq 1$.

- **Conjectured** : The super edge-magic deficiency of all 2-regular graphs of odd order is zero, excluding $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$.

J. Holden, D. McQuillan, and J. M. McQuillan, A conjecture on strong magic labelings of 2-regular graphs, *Discrete Math.*, **312**, (2009), 4130–4136.

# Results on 2-regular Graphs

Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that:

- $C_3 \cup C_n$ is SEMT iff $n \geq 6$ and $n$ is even.
- $C_4 \cup C_n$ is SEMT iff $n \geq 5$ and $n$ is odd.
- $C_5 \cup C_n$ is SEMT iff $n \geq 4$ and $n$ is even.
- $C_n \cup C_m$ is SEMT if $n$ is even and $m \geq \frac{n}{2} + 1$ is odd.

R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, A magical approach to some labeling conjectures, *Discuss. Math. Graph Theory*, **31**, (2011), 79–113.

## Results on 2-regular Graphs

- Let $G \cong \bigcup_{i=1}^{k} C_{n_i}$, $H \cong \bigcup_{i=1}^{k} (m, n_i) C_{[m,n_i]}$, and let $m$ be odd. If $G$ is SEMT, then $H$ is SEMT.
  - $(a, b)$ is the *greatest common divisor* of $a$ and $b$,
  - $[a, b]$ is the *least common multiple* of $a$ and $b$.

R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, Enlarging the classes of super edge-magic 2-regular graphs , *AKCE Int. J. Graphs Comb.*, **10** (2), (2013), 129–146.

## Results on 2-regular Graphs

A graph $G \cup tK_1$ is called *pseudo super edge-magic total* (PSEMT) if there exists a bijection $f : V(G \cup kK_1) \to \{1, 2, 3, \ldots |V(G)| + t\}$ such that the set $\{f(x) + f(y) : xy \in E(G)\} \cup \{2f(u) : u \in kK_1\}$ is a set of $|E(G)| + t$ consecutive integers. In such a case $f$ is called a *pseudo super edge-magic total labeling* (PSEMTL) of $G \cup kK_1$.

- Let $G \cong [\bigcup_{i=1}^{k} C_{n_i}] \cup tK_1$, $H \cong [\bigcup_{i=1}^{k} (m, n_i) C_{[m,n_i]}] \cup tC_m$, and let $m$ be odd. If $G$ is PSEMT, then $H$ is SEMT.
  - $(a, b)$ is the *greatest common divisor* of $a$ and $b$,
  - $[a, b]$ is the *least common multiple* of $a$ and $b$.

R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, Enlarging the classes of super edge-magic 2-regular graphs , *AKCE Int. J. Graphs Comb.*, **10** (2), (2013), 129–146.

# Results on 2-regular Graphs

Cichacz et al. [1] (2017) introduced a technique for constructing vertex-magic total labelings of 2-regular graphs and proved the following results, which contributes significantly to Holden et al. [2] conjecture.

- [1] If $G \cong \bigcup_{i=1}^{k} C_{n_i}$ is SEMT, then $H \cong \bigcup_{i=1}^{k} C_{mn_i}$ is SEMT for every odd $m$ .


[1]. S. Cichacz-Przenioslo, D. Froncek, and I. Singgih, Vertex magic total labelings of 2-regular graphs,*Discrete Math.*, **340** (1) (2017), 3117–3124. DOI: 10.1016/j.disc.2016.06.022

[2]. J. Holden, D. McQuillan, and J. M. McQuillan, A conjecture on strong magic labelings of 2-regular graphs, *Discrete Math.*, **312**, (2009), 4130 - 4136.

# Edge-Magic Deficiency

In 1970, Kotzig and Rosa proved that for every graph $G$ there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$.

- The *edge-magic deficiency* of a graph $G$, $\mu(G)$, is defined as $\mu(G) = min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}$.
- $\mu(G) \leq F_{p+2} - 2 - p - \frac{1}{2}p(p-1)$, where $p = |V(G)|$ and $F_p$ is the $p$th Fibonacci number.

A. Kotzig and A. Rosa, Magic valuation of finite graphs, *Canad. Math. Bull.*, **13** (4), (1970), 451– 461.

## Super Edge-Magic Deficiency

- Let $G$ be a graph and let

$$M(G) = min\{n \geq 0 : G \cup nK_1 \text{ is super edge-magic}\}.$$

The *super edge-magic deficiency* of a graph $G$, $\mu_s(G)$, is defined to be

$$\mu_s(G) = \begin{cases} min\ M(G), & \text{if } M(G) \neq \emptyset, \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases}$$

- $\mu_s(G)$ measure how "close" a graph to be a super edge-magic graph.
- $\mu(G) \leq \mu_s(G)$, for every graph $G$.

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 299–314.

# Example

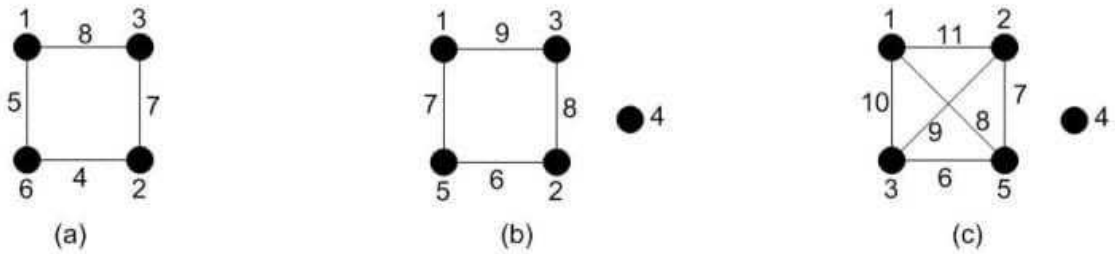


Figure: (a). $\mu(C_4) = 0$, (b). $\mu_s(C_4) = 1$, (c). $\mu(K_4) = \mu_s(K_4) = 1$

## Graph $G$ with $\mu_s(G) = +\infty$

- Let $G$ be a graph of size $q$ such that $\deg(v)$ is even for all $v \in V(G)$ and $q \equiv 2 \pmod 4$, then $\mu_s(G) = +\infty$.
  - $\mu_s(C_n) = +\infty$, if $n \equiv 2 \pmod 4$.

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 299–314.

## Graph $G$ with $\mu_s(G) = +\infty$

A set $X = \{x_1 < x_2 < \cdots < x_n\} \subseteq \mathbb{N}$ is a *well-spread set* (WS-set for short) if the sums $x_i + x_j$ for $i <$ are all different.

The *smallest span of pairwise sums of cardinality n*, denoted by $\rho^*(n)$,

$\rho^*(n) = min\{x_n + x_{n-1} - x_2 - x_1 : \{x_1 < x_2 < \cdots < x_n\}$ is a WS-set$\}$.

- The smallest span of pairwise sums of cardinality $n$, $\rho*(n)$ satisfies: $\rho^*(4) = 6$, $\rho^*(5) = 11$, $\rho^*(6) = 19$, $\rho^*(7) = 30$, $\rho^*(8) = 43$ and $\rho^*(n) > n^2 - 5n + 14$ for $n > 9$.

A. Kotzig, On well spread sets integers, *Publications du Centre de Recherches Mathematiques Universite de Montreal* **161**, (1972).

## Graph $G$ with $\mu_s(G) = +\infty$

- Let $G$ be a graph that contains the complete subgraph $K_n$. If $|E(G)| < \rho^*(n)$, then $\mu_s(G) = +\infty$.
  - For any positive integer $n$,

$$\mu_s(K_n) = \begin{cases} 0, & \text{if } n = 1, 2, 3, \\ 1, & \text{if } n = 4, \\ +\infty, & \text{for } \textit{otherwise}. \end{cases}$$

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 299–314.

# Super Edge-Magic Deficiency of Forests

- Let $F$ be a forest, then $\mu_s(F) \leq +\infty$.
- For every positive integer $n$,

$$\mu_s(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 299–314.

## Super Edge-Magic Deficiency of Forests

- For every two positive integers $m$ and $n$,

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd,} \\ & \text{or } m = 3 \text{ and } n \not\equiv 3 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

- For every two positive integers $m$ and $n$,

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{either } m \text{ is a multiple of } n+1, \\ & \text{or } n \text{ is a multiple of } m+1, \\ 1, & \text{otherwise.} \end{cases}$$

R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55**, (2005), 17–31.

# Super Edge-Magic Deficiency of Forests

- For every two positive integers $m$ and $n$,

$$\mu_s(P_m \cup P_n) = \begin{cases} 1, & \text{if } (m,n) \in \{(2,2),\ (3,3)\}, \\ 0, & \text{otherwise.} \end{cases}$$

- **Conjecture**: If $F$ is a forest with two components then $\mu_s(F) \leq 1$.

R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55**, (2005), 17–31.

# Super Edge-Magic Deficiency of Forests

A *banana tree* $BT(n_1, n_2, \ldots, n_k)$ is a tree obtained from the stars $K_{1,n_1}$, $K_{1,n_2}, \ldots,$ $K_{1,n_k}$ by joining a new vertex to a single leave of each star.

- Let $G \cong BT(n_1, n_2) \cup BT(m_1, m_2, \ldots, m_k)$. For $n_2 = 2k$, $n_1 \geq n_2 \geq m_1 \geq m_2 \geq \ldots, \geq m_k$ and $|V(G)| \equiv 2 \pmod 4$, then $\mu_s(G) \leq 1$.
- For every $n \geq 3$, $\mu_s(K_{1,n-1} \cup K_{1,n} \cup K_{1,5n-12}) = 0$.
- For $k \geq 3$ is odd, $n \geq 5$, and $m \geq 3$, $\mu_s(P_n \cup P_{n+k} \cup K_{1,m}) \leq n + \lfloor \frac{k}{2} \rfloor$.
- For $n \geq 4$ and $m \geq 3$, $\mu_s(P_{n-1} \cup P_n \cup K_{1,m}) \leq n - 1$.
- For $n \geq 3$ and $m \geq 3$, $\mu_s(P_n \cup P_{2n} \cup K_{1,m}) \leq \lfloor \frac{3n}{2} \rfloor - 1$.

A. Q. Baig, A. Ahmad, E. T. Baskoro, and R. Simanjuntak On the super edge-magic deficiency of forests , *Util. Math.*, **2**, (2011), 147–159.

# Super Edge-Magic Deficiency of Forests

Imran and Mukhtar (2017) proved that $\mu_s(G) = 0$ if

- $G \cong T(n, n, n, n+1) \cup K_{1,\frac{1}{2}(n-2)}$, $3 \leq n \equiv 1 \pmod 2$,
- $G \cong T(n, n-1, t, t+2) \cup K_{1,\frac{t}{2}}$, $t \geq n$ and $n, t \equiv 0 \pmod 2$,
- $G \cong T(n, n-1, t, t+2, 2t+4) \cup K_{1,t+1}$, $t \geq n \geq 4$ and $n, t \equiv 0 \pmod 2$,
- $G \cong T(n, n-1, t, t+2, 2t+4, 4t+8) \cup K_{1,2t+3}$, $t \geq n \geq 2$ and $n, l \equiv 0 \pmod 2$,
- $G \cong T(n, n-1, t, t+2, \ldots, t_r) \cup K_{1,\frac{t}{2}}$, $t \geq n \geq 2$, $n, t \equiv 0 \pmod 2$, and $4 \leq t_r = 2^{r-4}(t+2)r$,

where $T(n_1, n_2, \ldots, n_r)$ is a graph obtained by replacing each edge of a star $K_{1,n}$ by a path of length $n_1, n_2, \ldots, n_r$, respectively.

M. Imran and A. Mukhtar, Super edge-magic total labeling of forests consisting of stars and subdivided stars, *Int. J. of Math. and soft Comput.*, **2**, (2017), 1–14.

# Super Edge-Magic Deficiency of $K_{n,m}$

- For every two positive integers $m$ and $n$, $\mu_s(K_{n,m}) \leq (n-1)(m-1)$.
- For any positive integer $m$, $\mu_s(K_{2,m}) = m - 1$ .
- **Conjecture** : *For every two positive integers* $m$ and $n$, $\mu_s(K_{n,m}) = (n-1)(m-1)$

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 299–314.

# Super Edge-Magic Deficiency of $K_{n,m}$

- For $n = 2, 3$ and 4 and for any positive integers $m$, $\mu_s(K_{n,m}) = (n-1)(m-1)$ [1].
- For every two positive integers $m$ and $n$, $\mu_s(K_{n,m}) = (n-1)(m-1)$ [2].


[1]. S. M. Hegde, S. Shetty and P. Shankaran, Further results on the super edge-magic deficiency of graphs, *Ars Combin.*, **99**, (2011), 487–502.

[2]. R. Ichishima and A. Oshima, On the super edge-magic deficiency and $\alpha$-valuations of graphs, *J. Indones. Math. Soc.*, Special Edition, (2011), 59–69.

# Super Edge-Magic Deficiency of $tK_{1,m}$

Figueroa-Centeno et al. (2005) [1]:

- For all positve integers $t$ and $n$ such that $t$ is odd, $\mu_s(tK_{1,n}) = 1$.

Baskoro and AAGN (2003) [2]:

- For all even integers $t \geq 4$, $\mu_s(tK_{1,2}) = 0$.


[1]. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On edge-magic labeling of certain disjoint unions of graphs , *Australas J. Combin.*, **32**, (2005), 225–242.

[2]. E. T. Baskoro and Ngurah, On super edge-magic total labeling of $nP_3$, *Bull. Inst. Combin. Appl.*, **37**, (2003), 82–87.

## Super Edge-Magic Deficiency of $tK_{1,m}$

- For all positive integers $t$ and $m$ such that $t$ is even, $\mu_s(tK_{1,m}) \leq 1$.
- For every positive integer $m$, $\mu_s(2K_{1,m}) = 1$.
- For all positive integers $t$ and $m$ such that $t \equiv 2 \pmod 4$ and $m$ is odd, $\mu_s(tK_{1,m}) = 1$.
- For every positive integer $t$,

$$\mu_s(tK_{1,3}) = \begin{cases} 0, & \text{if } t \equiv 4 \pmod 8 \textit{ or } t \textit{ is odd}, \\ 1, & \text{if } t \equiv 2 \pmod 4. \end{cases}$$

- **Open Problem**: *For even $t \geq 4$ and $m \geq 3$, determine the exact value of $\mu_s(tK_{1,m})$.*

R. Ichishima and A. Oshima, On the super edge-magic deficiency and $\alpha$-valuations of graphs, *J. Indones. Math. Soc.*, Special Edition, (2011), 59 - 69.

## Super Edge-Magic Deficiency of $tK_{n,m}$

Simanjuntak, Baskoro, Uttunggadewa and AAGN (2008)[1] :

- For all integers $t$, $n$ and $m$, with $t \geq 1$, $n \geq 4$ and $m \geq 4$, $\mu_s(tK_{n,m}) \leq t(nm - n - m) + 1$.

Ichishima and Oshima (2011)[2] :

- For all integers $t$, $n$ and $m$, with $t \geq 1$, $n \geq 2$ and $m \geq 2$, $\mu_s(tK_{n,m}) \leq t(nm - n - m) + 1$.

- **Conjecture**: For all integers $t$, $n$ and $m$, with $t \geq 1$, $n \geq 2$ and $m \geq 2$, $\mu_s(tK_{n,m}) = t(nm - n - m) + 1$.


[1]. A.A.G. Ngurah, E.T. Baskoro, R. Simanjuntak, and S. Uttunggadewa, On the super edge-magic strength and deficiency of graphs, *Comp. Geometry and Graph Theory*, LNCS, **4535** (2008), 144–154.

[2]. R. Ichishima and A. Oshima, On the super edge-magic deficiency and $\alpha$-valuations of graphs, *J. Indones. Math. Soc.*, Special Edition, (2011), 59–69.

# Super Edge-Magic Deficiency of $Q_n$

- For $n \geq 4$, $(n-4)2^{n-2} + 3 \leq \mu_s(Q_n) \leq (n-2)2^{n-1} - 4$.
- **Open Problem**. *Determine the exact value of* $\mu_s(Q_n)$.

R. Ichishima and A. Oshima, On the super edge-magic deficiency and $\alpha$-valuations of graphs, *J. Indones. Math. Soc.*, Special Edition, (2011), 59–69.

# Super Edge-Magic Deficiency of 2-regular Graphs

- For every integer $n \geq 3$,

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \equiv 1 \text{ or } 3 \pmod 4, \\ 1, & \text{if } n \equiv 0 \pmod 4, \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Electron. Notes Discrete Math.*, **11**, (2002), 299–314.

# Super Edge-Magic Deficiency of 2-regular Graphs

- For every integer $n \geq 3$,

$$\mu_s(2C_n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ +\infty, & \text{if } n \text{ is odd.} \end{cases}$$

- For every integer $n \geq 3$,

$$\mu_s(3C_n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \equiv 0 \pmod 4, \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

- For $n \equiv 0 \pmod 4$, $\mu_s(4C_n) = 1$.

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55**, (2005), 17–31.

# Super Edge-Magic Deficiency of 2-regular Graphs

- **Conjecture**: For every integers $m \geq 1$ and $n \geq 3$,

$$\mu_s(mC_n) = \begin{cases} 0, & \text{if } mn \text{ is odd}, \\ 1, & \text{if } mn \equiv 0 \pmod 4, \\ +\infty, & \text{if } mn \equiv 2 \pmod 4. \end{cases}$$

R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55**, (2005), 17–31.

## Super Edge-Magic Deficiency of 2-regular Graphs

Holden et al. proved that, for some integer $t$, $C_5 \cup (2t)C_3$, $C_4 \cup (2t-1)C_3$, and $C_7 \cup (2t)C_3$, are strong vertex-magic, which is in fact equivalent to saying that they are super edge-magic. In other words, they proved that

- $\mu_s(C_5 \cup (2t)C_3) = 0$ for each integer $t \geq 3$.
- $\mu_s(C_4 \cup (2t-1)C_3) = 0$ for each integer $t \geq 3$.
- $\mu_s(C_7 \cup (2t)C_3) = 0$ for each integer $t \geq 1$.

- **Conjectured** : The super edge-magic deficiency of all 2-regular graphs of odd order is zero, excluding $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$.

J. Holden, D. McQuillan, and J. M. McQuillan, A conjecture on strong magic labelings of 2-regular graphs, *Discrete Math.*, **312**, (2009), 4130–4136.

# Super Edge-Magic Deficiency of 2-regular Graphs

Figueroa-Centeno, Ichishima, and Muntaner-Batle [1] proved that:

- $\mu_s(C_3 \cup C_n) = 0$ iff $n \geq 6$ and $n$ is even.
- $\mu_s(C_4 \cup C_n) = 0$ iff $n \geq 5$ and $n$ is odd.
- $\mu_s(C_5 \cup C_n) = 0$ iff $n \geq 4$ and $n$ is even.
- $\mu_s(C_n \cup C_m) = 0$ if $n$ is even and $m \geq \frac{n}{2} + 1$ is odd.

- Ichishima and Oshima [2] investigated the super edge-magic deficiency of 2-regular graphs $C_m \cup C_n$ for $m = 3, 4, 5, 7$ and any $n$.

[1] R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, A magical approach to some labeling conjectures, *Discuss. Math. Graph Theory*, **31**, (2011), 79–113.

[2] R. Ichishima and A. Oshima, On the super edge-magic deficiency of 2-regular graphs with two components, *Ars Combin.*, **129** (2016), 437 – 447.

# Super Edge-Magic Deficiency of 2-regular Graphs

- Let $G \cong \bigcup_{i=1}^{k} C_{n_i}$, $H \cong \bigcup_{i=1}^{k} (m, n_i) C_{[m,n_i]}$, and let $m$ be odd. If $\mu_s(G) = 0$, then $\mu_s(H) = 0$.
  - $(a, b)$ is the *greatest common divisor* of $a$ and $b$,
  - $[a, b]$ is the *least common multiple* of $a$ and $b$.

R. Ichishima, F. A. Muntaner-Batle, and A. Oshima, Enlarging the classes of super edge-magic 2-regular graphs , *AKCE Int. J. Graphs Comb.*, **10** (2), (2013), 129–146.

# Super Edge-Magic Deficiency of 2-regular Graphs

Cichacz et al. [1] (2017) introduced a technique for constructing vertex-magic total labelings of 2-regular graphs and proved the following results, which contributes significantly to Holden et al. [2] conjecture.

- [1] If $\mu_s(\bigcup_{i=1}^{k} C_{n_i}) = 0$, then $\mu_s(\bigcup_{i=1}^{k} C_{mn_i}) = 0$ for every odd $m$.


[1]. S. Cichacz-Przenioslo, D. Froncek, and I. Singgih, Vertex magic total labelings of 2-regular graphs, *Discrete Math.*, **340** (1) (2017), 3117–3124. DOI: 10.1016/j.disc.2016.06.022

[2]. J. Holden, D. McQuillan, and J. M. McQuillan, A conjecture on strong magic labelings of 2-regular graphs, *Discrete Math.*, **312**, (2009), 4130 - 4136.

# Super Edge-Magic Deficiency of Join Product Graphs

*Join product* of two vertex disjoin graphs $G$ and $H$, $G + H$, is their graph union with additional edges that connect all vertices of $G$ to each vertex of $H$.

- For any integers $n, m \geq 3$,
  $\lceil \frac{1}{2}(n-2)(m-1) \rceil \leq \mu_s(P_n + mK_1) \leq (n-1)(m-1) - 1.$
- For any integers $n, m \geq 2$,
  $\lceil \frac{1}{2}(n-1)(m-1) \rceil \leq \mu_s(K_{1,n} + mK_1) \leq n(m-1) - 1.$
- For any integers $n \geq 3$ and $m \geq 2$,
  $\mu_s(C_n + mK_1) \geq \lfloor \frac{1}{2}(m+1)n \rfloor - (n+m) + 2.$
- For any integer $m \geq 2$ and odd integer $n \geq 3$,
  $\mu_s(C_n + mK_1) \leq mn - (n+m) + 1.$

A. A. G. Ngurah and R. Simanjuntak, Super edge-magic of join product graphs, *Util. Math.*, **105** (2017), 279 – 289.

# Super Edge-Magic Deficiency of Join Product Graphs

- Let $G$ be a super edge-magic graph of order $p$ and size $q \geq 1$ with a super edge-magic labeling $f$. For any integer $m \geq 1$,

$$\mu_s(G + mK_1) \leq \begin{cases} p + 1 - min(S), & \text{if } m = 1, \\ (m-2)(p-1) + (q-1), & \text{if } m \geq 2, \end{cases}$$

where $S = \{f(x) + f(y) : xy \in E(G)\}$.

**Open Problem**: Find a better upper bound of $\mu_s(G + mK_1)$ when $G$ is a super edge-magic graph.

A. A. G. Ngurah and R. Simanjuntak, Super edge-magic labelings: deficiency and maximality, *EJGTA*, **5** (2) (2017), 212 – 220.

# Super Edge-Magic Deficiency of Join Product Graphs

- [1] Let $G$ be a graph with no cycle and isolated vertices. If $\mu_s(G + K_1) = 0$, then $G$ is a tree or a forest.
  - $\mu_s([K_{1,n} \cup K_2] + K_1) = 0$ iff $n = 2$. [1]
  - $\mu_s([P_n \cup K_2] + K_1) = 0$ iff $n = 3, 4, 5$. [1]
  - $\mu_s(\mathrm{DS}_n + K_1) = 0$, $n \geq 1$, where $\mathrm{DS}_n$ is a double star. [1]
  - $\mu_s(F_n = P_n + K_1) = 0$ iff $1 \leq n \leq 6$. [2]
  - $\mu_s(C_3^n = nK_2 + K_1) = 0$ iff $n = 3, 4, 5, 7$. [3]


[1]. A. A. G. Ngurah and R. Simanjuntak, Super edge-magic deficiency of join prouduct and chain graphs, EJGTA, **7** (1) (2019), 157 – 167.

[2]. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001), 153 - 168.

[3]. Slamin, M. Baca, Y. Lin, M. Miller and R. Simanjuntak Edge-magic total labelings of wheels, fans and frienship graphs, *Bulletin of The ICA*, **35** (2002), 89–98.

# Super Edge-Magic Deficiency of Join Product Graphs

- Let $G$ be a tree of order $n \geq 7$ and $H \cong G + K_1$. If $\mu_s(H) = 0$ then either $K_3 \cup K_{1,3}$ or $2K_{1,3}$ is a subgraph of $H$.
- If $G$ is any tree of order $|V(G)| \leq 6$ excluding $G_1$ then $\mu_s(G + K_1) = 0$.

$G_1$

- $\mu_s(G_1 + K_1) = 1$.

A. A. G. Ngurah and R. Simanjuntak, Super edge-magic deficiency of join prouduct and chain graphs, EJGTA, **7** (1) (2019), 157 – 167.

# Super Edge-Magic Deficiency of Join Product Graphs

- [1] Let $G$ be a tree and $m \geq 2$ be an integer.

$$\mu_s(G + mK_1) = 0 \text{ iff } G = P_2.$$

- [1] Let $G$ be a tree of order $n \geq 3$ . For every positive integer $m \geq 2$,

$$\mu_s(G + mK_1) \geq \lfloor \frac{(m-1)(n-2)+1}{2} \rfloor.$$

  - $\mu_s(P_4 + mK_1) = m - 1$ and $\mu_s(P_6 + mK_1) = 2(m-1)$. [2]
  - $\mu_s(P_n + 2K_1) = \frac{n-2}{2}$ for any even integer $n \geq 2$. [3]


[1]. A. A. G. Ngurah and R. Simanjuntak, Super edge-magic deficiency of join prouduct and chain graphs, EJGTA, **7** (1) (2019), 157 – 167.

[2]. Ngrh and R. Simanjuntak, Super edge-magic of join product graphs, *Util. Math.*, **105** (2017), 279 – 289.

[3]. Ngrh, E. T. baskoro, R. Simanjuntak, and S. Uttunggadewa On super edge-magic strength and deficiency of graphs, *Computational Geometry and Graph Theory*, LNCS, **4535** (2008), 144 - 154.

# Super Edge-Magic Deficiency of Join Product Graphs

- Let $G$ be a super edge-magic graph of order $p$ and size $q \geq 1$ with a super edge-magic labeling $f$. For any integer $m \geq 1$,

$$\mu_s(G + mK_1) \leq \begin{cases} p + 1 - min(S), & \text{if } m = 1, \\ (m-2)(p-1) + (q-1), & \text{if } m \geq 2, \end{cases}$$

where $S = \{f(x) + f(y) : xy \in E(G)\}$.

**Open Problem**: Find a better upper bound of $\mu_s(G + mK_1)$ when $G$ is a super edge-magic graph.

A. A. G. Ngurah and R. Simanjuntak, Super edge-magic labelings: deficiency and maximality, *EJGTA*, **5** (2) (2017), 212 – 220.

# Super Edge-Magic Deficiency of Join Chain Graphs

- A *chain graph* is a graph with blocks $B_1, B_2, B_3, \cdots, B_k$ such that for every $i$, $B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cut-vertex graph is a path.
- A chain graph with blocks $B_1, B_2, B_3, \cdots, B_k$ is denoted by $C[B_1, B_2, \cdots, B_k]$.
- If $B_1 = B_2 = \ldots = B_t \cong B$ then $C[B_1, B_2, \cdots, B_k]$ is denoted by $C[B^{(t)}, B_{t+1}, \ldots, B_k]$.
- If for every $i$, $B_i \cong H$ then $C[B_1, B_2, \cdots, B_k]$ is denoted by $kH$-path.
- Let $c_1, c_2, \ldots, c_{k-1}$ be the consecutive cut vertices of $C[B_1, B_2, \cdots, B_k]$. The *string* of $C[B_1, B_2, \cdots, B_k]$ is $(k-2)$-tuple $(d_1, d_2, \ldots, d_{k-2})$ where $d_i$ is the distance between $c_i$ dan $c_{i+1}$, $1 \leq i \leq k-2$.

# Super Edge-Magic Deficiency of Chain Graphs

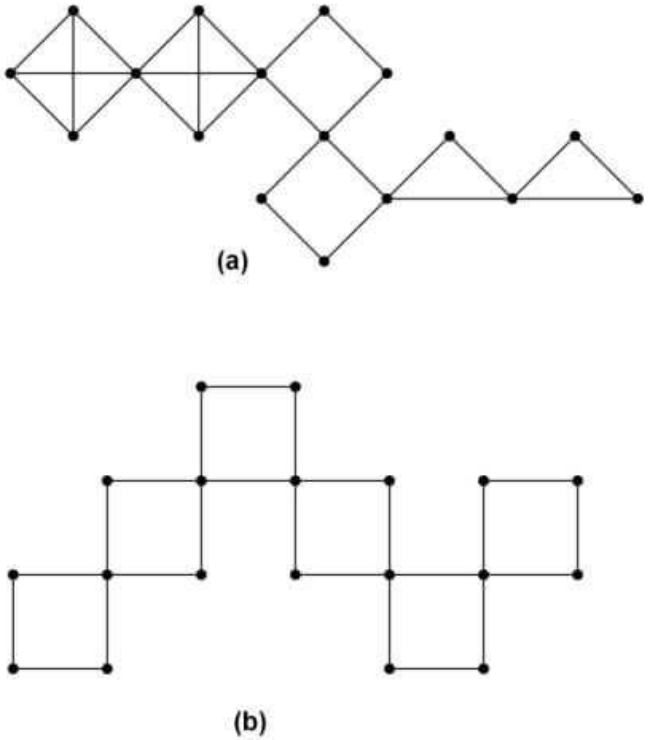


Figure: (a) The chain graph $C[K_4^{(2)}, C_4^{(2)}, K_3^{(2)}]$ with string $(1, 1, 1, 1)$.
(b). The chain graph $6C_4$-path with string $(2, 1, 2, 1)$.

# Super Edge-Magic Deficiency of Chain Graphs

- If $G \cong kK_3$-path then, for any integer $k \geq 3$,

$$\mu_s(G) = \begin{cases} 0, & \text{if } k \equiv 0,\ 1 \pmod 4, \\ +\infty, & \text{if } k \equiv 2 \pmod 4, \end{cases}$$

  and $\mu_s(G) \leq k - 1$, if $k \equiv 3 \pmod 4$.

- If $G \cong kK_4$-path then, for any integer $k \geq 3$, $\mu_s(G) = 1$.

A. A. G. Ngurah, R. Simanjuntak, and ETB, On the super edge-magic deficiency of graphs, *Australas. J. Combin.*, **40** (2008), 3–14.

# Super Edge-Magic Deficiency of Chain Graphs

- A *diagonal ladder*, $\mathrm{DL}_m$, is a graph obtained from the ladder $L_m \cong P_m \times P_2$ by adding two diagonals in each rectangle of $L_m$.
- $\mu_s(\mathrm{DL}_m) = \lfloor \frac{m}{2} \rfloor$, for every $m \geq 2$. [1]
- Let $H \cong k\mathrm{DL}_m$-path with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{1, 2, \ldots, m-1\}$.

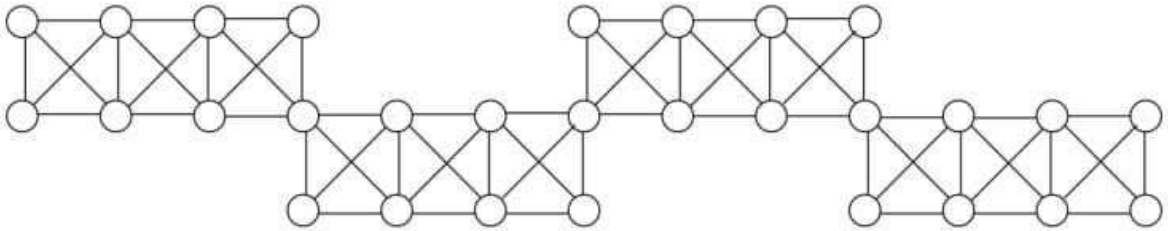

Figure: A $4\mathrm{DL}_4$-path with string $(3, 3)$.

[1]. A. Ahmad, M. F. Nadeem, M. Javaid, and R. Hasni, On the super edge-magic deficiency of some families related to ladder graphs, *Australas. J. Combin.*, **51** (2011), 201–208.

# Super Edge-Magic Deficiency of Chain Graphs

- For any integers $k \geq 3$ and $m \geq 2$, $\mu_s(H) \geq \lfloor \frac{1}{2}(m-2)k+1 \rfloor$.
- For any integers $k \geq 3$ and $m \geq 2$, if $H$ has string $(m-1, \ldots, m-1)$ then $\mu_s(H) = \lfloor \frac{1}{2}(m-2)k+1 \rfloor$.

A. A. G. Nugrah, On the (super) edge-magic deficiency of chain graphs, *J. Combin. Math. Combin. Comput.*, **103**, (2017), 225 – 236.

# Super Edge-Magic Deficiency of Chain Graphs

- Let $F \cong C[K_4^{(k)}, \mathrm{DL}_m, K_4^{(n)}]$ with string $(1^{(k-1)}, d, 1^{(n-1)})$, where $d \in \{1, 2, 3, \ldots, m-1\}$.

  - For any integers $k, n \geq 1$ and $m \geq 2$, $\mu_s(F) \geq \lfloor \frac{m}{2} \rfloor$.
  - For any integers $k, n \geq 1$ and $m \geq 2$, if $F$ has string $(1^{(k-1)}, m-1, 1^{(n-1)})$ then $\mu_s(F) = \lfloor \frac{m}{2} \rfloor$.

A. A. G. Nugrah, On the (super) edge-magic deficiency of chain graphs, *J. Combin. Math. Combin. Comput.*, **103**, (2017), 225 – 236.

# Super Edge-Magic Deficiency of Chain Graphs

- A *triangle ladder*, $\mathrm{TL}_m$, is a graph obtained from the ladder $L_m \cong P_m \times P_2$ by adding a single diagonal in each rectangle of $L_m$.
- $\mu_s(\mathrm{TL}_m) = 0$, for every $m \geq 2$. [1]
- For $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = \mathrm{TL}_m$ when $j$ is odd and $B_j = \mathrm{DL}_m$ when $j$ is even.

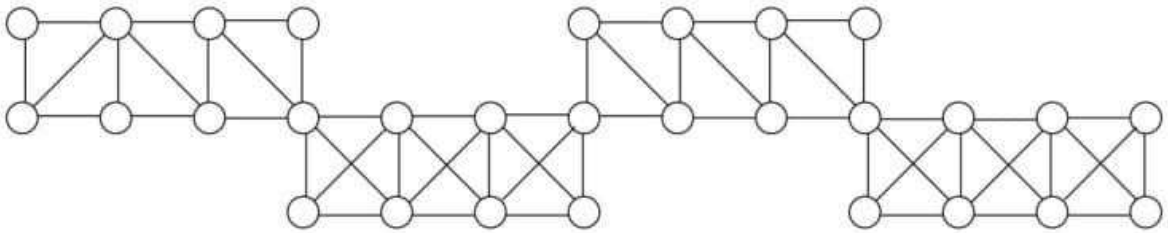

Figure: A chain graph $C[\mathrm{TL}_4, \mathrm{DL}_4, \mathrm{TL}_4, \mathrm{DL}_4]$ with string $(3, 4)$.

[1]. A. A. G. Ngurah and R. Simanjuntak, Super edge-magic labelings: deficiency and maximality, *EJGTA*,**5** (2) (2017), 212 – 220.

## Super Edge-Magic Deficiency of Chain Graphs

- For any integer $m \geq 3$,

$$\mu_s(G) \geq \begin{cases} \lfloor \frac{1}{4}k(m-3) \rfloor + 1, & \text{if } k \text{ is even,} \\ \lfloor \frac{1}{4}(k(m-3)-(m-1)) \rfloor + 1, & \text{if } k \text{ is odd.} \end{cases}$$

- If $G$ has string $(m-1, d_1, m-1, d_2, m-1 \ldots, d_{\frac{1}{2}(k-3)}, m-1)$ when $k$ is odd or $(m-1, d_1, m-1, d_2, \ldots, m-1, d_{\frac{1}{2}(k-2)})$ when $k$ is even, where $d_1, d_2, \ldots, d_{\lfloor \frac{1}{2}(k-2) \rfloor} \in \{m-1, m\}$, then for any odd integer $m \geq 3$,

$$\mu_s(G) = \begin{cases} \frac{1}{4}k(m-3)+1, & \text{if } k \text{ is even,} \\ \frac{1}{4}(k-1)(m-3), & \text{if } k \text{ is odd.} \end{cases}$$

A. A. G. Ngurah and Adiwijaya, New results on the (super) edge-magic deficiency of chain graphs, *Int. J. Math. and Mathematical Sciences*, (2017), Article ID 5156974.

# Super Edge-Magic Deficiency of Chain Graphs

- Let $H \cong C[K_4^{(p)}, \mathrm{TL}_m, K_4^{(q)}]$ with string $(1^{(p-1)}, d, 1^{(q-1)})$, where $d \in \{m-1, m\}$. For any integers $p, q \geq 1$ and $m \geq 2$, $\mu_s(H) = 0$.

A. A. G. Ngurah and Adiwijaya, New results on the (super) edge-magic deficiency of chain graphs, *Int. J. Math. and Mathematical Sciences*, (2017), Article ID 5156974.

# Super Edge-Magic Deficiency of Chain Graphs

• For $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = \mathrm{TL}_n$ when $j$ is odd and $B_j = \mathrm{DL}_m$ when $j$ is even, where $n$ is not necessarily equal to $m$.

- If $G$ has string $(m-1, d_1, m-1, d_2, m-1 \ldots, d_{\frac{1}{2}(k-3)}, m-1)$ when $k$ is odd or $(m-1, d_1, m-1, d_2, \ldots, m-1, d_{\frac{1}{2}(k-2)})$ when $k$ is even, where $d_1, d_2, \ldots, d_{\lfloor \frac{1}{2}(k-2) \rfloor} \in \{n-1, n\}$, then for any integers $n \geq 2$ and $m \geq 3$ such that $m$ is odd,

$$\mu_s(G) = \begin{cases} \frac{1}{4}k(m-3)+1, & \text{if } k \text{ is even,} \\ \frac{1}{4}(k-1)(m-3), & \text{if } k \text{ is odd.} \end{cases}$$

A. A. G. Ngurah and R. Simanjuntak, Super edge-magic deficiency of join prouduct and chain graphs, EJGTA, **7** (1) (2019), 157 – 167.

# Super Edge-Magic Deficiency of Chain Graphs

- For every integer $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = \mathrm{DL}_m$ when $j$ is odd and $B_j = \mathrm{TL}_n$ when $j$ is even.

  - If $G$ has string $(d_1, m-1, d_2, m-1, \ldots, m-1, d_{\frac{1}{2}(k-2)})$, where $d_1, d_2, \ldots, d_{\frac{1}{2}(k-2)} \in \{n-1, n\}$, then for any integers $n \geq 2$, $k \geq 3$ and $m \geq 3$ such that $k$ and $m$ are odd,

$$\frac{1}{4}(k+1)(m+1) - k \leq \mu_s(G) \leq \frac{1}{4}(k+1)(m+1) - (k-1).$$

A. A. G. Ngurah and R. Simanjuntak, Super edge-magic deficiency of join prouduct and chain graphs, EJGTA, **7** (1) (2019), 157 – 167.

## References


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THANK YOU FOR YOUR ATTENTION